Analysis of a Compressible Laminar Boundary Layer on a Yawed Cone

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The boundary-layer equations for a cone at incidence, involving azimuthal diffusion terms, are written as a system of nonlinear elliptic equations by means of the local similarity assumption. Several iterative methods are discussed for a single model elliptic equation. Among them, the line Gauss-Seidel method was preferable. A second-order scheme of discretization, unconditionally stable, is described and was used to solve the model equation as well as the system of boundary-layer equations. The solutions exhibit an "ellipticity region" close to the leeward plane. The abscissa effect on the length of this region and on the behavior of the main properties of the boundary layer is emphasized. The reliability of these predictions is discussed with reference to experimental results.

Nomenclature

a,b,c	= auxiliary variables (Sec. IV)
C	$=\rho\mu/(\rho_e\mu_e)$ in Eqs. (1-3)
B,C,D,E,F	= coefficients of Eq. (10)
DU	= auxiliary difference variable (Sec. IV)
i	= incidence
j,k	= finite-difference grid point in X, Y directions
K',M',Q'	= external flow parameters
M_∞	= Mach number at infinity
p,q	= auxiliary variables (Sec. IV)
P,Q,R,S	= coefficients of Eq. (6)
Pr	= Prandtl number
r	= geometric progression ratio
Re	= unitary local Reynolds number
T	= temperature
u,v,w	= dimensionless velocity in x, y, ϕ directions
x,y,ϕ	= coordinate system (Fig. 1)
X,Y	= auxiliary coordinates (Secs. III and IV)
α,β	= coefficients of Eq. (6)
δ	= displacement thickness
ϵ	= required precision
η	= Lees-Dorodnitsyn similarity variable
ϕ	= azimuthal angle
θ	$=\phi\sin\theta_c$
θ_c	= cone half-angle
μ	= viscosity (Sutherland law)
ρ	= density
$\Delta\phi_{\min}$	= minimum step size near $\phi=\pi$
Subscripts	
e	= external conditions
p	= wall conditions
Superscripts	
$\overline{}$	= conditions determined at the previous iteration

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I. Introduction

IT has been shown by several authors[1-6] that the Prandtl boundary-layer equations cannot always represent adequately the flow at the wall, on the leeside of a cone at incidence in a supersonic flow. These equations are parabolic in the ϕ direction and do not permit the boundary conditions in the plane of symmetry on this side ($\phi=\pi$) always to be satisfied. Thus, it should be necessary to maintain the elliptic character of the full Navier-Stokes equations, in keeping at least the second azimuthal derivatives in the momentum and energy equations. The importance of this requirement already has been pointed out by Lin and Rubin.[4] Such a consideration is supported by experimental evidence, the measurements[7,8] showing up large azimuthal gradients around $\phi=\pi$. The aim of the investigation described herein is not only to try and solve the new boundary-layer equations but also to tackle the problem of integrating equations elliptic in character which may assume a character practically parabolic inside the domain of integration.

II. Governing Equations: Similarity Assumption

For the flowfield around a cone at incidence, the Eulerian equations permit one to satisfy the physical boundary conditions everywhere except at the wall. (In particular, the symmetry conditions are verified on both sides: $\phi=0$ and $\phi=\pi$.) Thus, a boundary-layer model is to be used only in the vicinity of this wall. Such a model was used in Ref. 2 with the first-order approximation. It follows that $y\cos\theta_c$ is assumed to be small compared with $x\sin\theta_c$, making these equations invalid for problems with very small cone angle θ_c and for problems

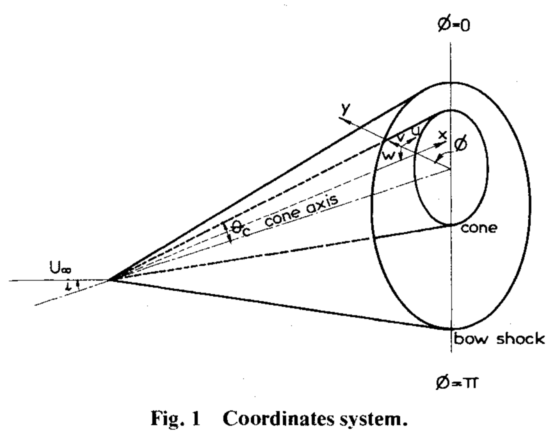

Fig. 1 Coordinates system.

very close to the tip on a sharp cone. In addition, a conical inviscid flow and a small incidence i (or small crossflow) were assumed in order to reduce the y-momentum equation to $(\partial p/\partial y)=0$. In this way, the Prandtl boundary-layer equations appeared to be self-similar and were derived using the Lees-Dorodnitsyn variable

$$\eta = Re^{1/2} x^{-1/2} \int_0^y \left(\frac{\rho}{\rho_e}\right) dy$$

However, the numerical investigation of these equations carried out in Ref. 2 showed that inside the boundary layer the symmetry conditions at $\phi=\pi$ cannot be satisfied for all incidences. The same result is obtained when parabolic similarity for boundary layer is not prescribed, as in the analysis given by Popinski and Davis.[6] In fact, the results obtained in Ref. 2 for incidences $0<i<i_{sep}$ (where i_{sep} denotes incidence at which the separation occurs due to an adverse pressure gradient in the ϕ direction) showed that (except for very small incidences, i.e., i/θ_c close to 0.1) the ϕ component of velocity w and the ϕ gradients of the temperature $\partial T/\partial\phi$ and of the longitudinal component of the velocity $\partial u/\partial\phi$ do not vanish at $\phi=\pi$, as would be required by symmetry conditions. The w velocity and the ϕ gradients then are discontinuous at $\phi=\pi$, as also was found by Murdock.[3] This indicates a defect in the usual parabolic boundary-layer model for which the governing equations are of lower degree in the ϕ direction than the full Navier-Stokes equations, as pointed out by Lin and Rubin.[4] This implies to maintain, in the ϕ direction, the elliptic characters of the Navier-Stokes equations and thus to produce a boundary region formulation near $\phi=\pi$ in which $(\partial w/\partial\phi)$, $(\partial^2 w/\partial\phi^2)$, $(\partial^2 w/\partial\phi\partial y)$, $(\partial^2 u/\partial c^2)$, and $(\partial^2 T/\partial\phi^2)$ are important. In this way, the following governing equations, valid for the nonseparated boundary layer, can be derived from the Navier-Stokes equations, retaining, as in Ref. 9 the viscous terms that involve these derivatives, and assuming constant pressure across the boundary layer. If follows that

$$-\left[\frac{4}{3}\frac{T^2}{Rex}Cw_{\theta\theta}+Cw_{\eta\eta}\right]+(v-C_\eta)w_\eta+ww_\theta+uw+K'w^2$$
$$=(1+M')K'T \qquad (1)$$

$$-\left[\frac{T^2}{Rex}Cu_{\theta\theta}+Cu_{\eta\eta}\right]+(v-C_\eta)u_\eta+wu_\theta+K'wu=w^2 \qquad (2)$$

$$-\left[\frac{T^2}{Rex}\frac{C}{Pr}T_{\theta\theta}+\frac{C}{Pr}T_{\eta\eta}\right]+\left(v-\left(\frac{C}{Pr}\right)_\eta\right)T_\eta+wT_\theta$$
$$=\frac{-C(\gamma-1)M_e^2}{1+K'^2}(u_\eta^2+w_\eta^2) \qquad (3)$$

$$v_\eta=-\frac{3}{2}u-w\left[\frac{Q'+K'}{2}\right]-w_\theta \qquad (4)$$

where T is the temperature referred to his external flow value, and u and w are the longitudinal and azimuthal components of the velocity referred to U_e:

$$Re=\rho_e U_e/\mu_e, \quad w=W/U_e, \quad K'=W_e/U_e$$

$$M'=(1/U_e)\partial W_e/\partial\theta,$$

$$Q'=\frac{1}{\rho_e\mu_e}\frac{\partial\rho_e\mu_e}{\partial\theta}$$

The boundary conditions considered in this study are the following:

when $\theta=0$ or $\theta=\pi\sin\theta_c$: $w=0, \quad u_\theta=0, \quad T_\theta=0$ (5a)

when $\eta=0$: $w=u=v=0, \quad T=T_p=\text{const}$ (5b)

when $\eta\to\infty$: $w\to K', \quad u\to 1, \quad T\to 1$ (5c)

In addition, in the more general case (with separation), the governing equations must probably include cross-derivative terms as in Ref. 4 or 5. But these additional terms do not modify the modeling of the linearized and uncoupled equations to be used in the following.

III. Elliptic Model Equation

Convergence Conditions

As a first approach, we consider the iterative resolution of a model equation of the same kind as Eqs. (1-3), that is,

$$-[\alpha U_{YY}+\beta U_{XX}]+PU_X+QU_Y+RU=S \qquad (6)$$

such that $\alpha\geq 0$, $\beta\geq 0$, and $R\geq 0$.

We intend to use a Jacobi generalized method, that is, one among the following methods: Jacobi (J), Line-Jacobi (LJ), Gauss-Seidel (GS), or Line-Gauss-Seidel (LGS). For such methods, we can derive such general conditions sufficient to insure the convergence of several schemes using a five-point approximation. When discretizing this model equation, we put the system of difference equations in a matrix form, $AU=S$; then we try and obtain some schemes of discretization for which the matrix A has particular properties (V matrix) like the Stieltjes matrices discussed by Varga.[10] We shall call V matrix a real matrix A $(n\times n)$(nonsingular) in which the entries verify, for any $1\leq i\leq n$, the following conditions:

$$A_{ij}\leq 0 \text{ for } i\neq j, \quad A_{ii}>0, \quad A_{ii}>\sum_{j\neq i}|A_{ij}| \qquad (7)$$

We can show the following, as in Ref. 11, as a direct application of the theorems given by Varga:

1) If A is a V matrix, then any splitting of $A=M-N$, where M is a nonsingular matrix obtained by setting certain off-diagonal entries of the matrix A to zero, is "regular" (with the definition given by Varga,[10] and the spectral radius $\rho(M^{-1}N)<1$.

2) If A is a V matrix and $(M-N)$ is a regular splitting of A, then, $N_1\geq N_2\geq 0$, equality excluded, we have

$$0<\rho(M_2^{-1}N_2)<\rho(M_1^{-1}N_1)<1$$

It follows that, for a V matrix, the splittings corresponding to the algorithms (J), (LJ), (GS), (LGS) are convergent (property 1)and that, among them, the method (LGS) is the most quickly convergent (property 2).

Linear Model Equation

We consider Eq. (6), defined on a rectangular domain (X_0, X_{n+1}), (Y_0, Y_{m+1}) with the following boundary conditions:

$$U(X_0,Y)=U_0(Y); \quad U(X_{n+1},Y)=U_{n+1}(Y) \qquad (8)$$

$$U(X,Y_0)=W_0(X); \quad U(X,Y_{m+1})=W_{m+1}(X) \qquad (9)$$

We use a five-point approximation to transform Eq. (6) into the following system of difference equations valid for all interior point (X_j, Y_k):

$$-EU_{j,k-1}-BU_{j-1,k}+DU_{j,k}-CU_{j+1,k}-FU_{j,k+1}=S \qquad (10)$$

where D stands for D_{jk}, B for B_{jk}, etc., and

$$D=B+C+E+F+R$$

For the linear model equation (6) with the boundary conditions (8) and (9), the convergence conditions (7) can be derived easily into

$$B\geq 0, \quad C\geq 0, \quad E\geq 0, \quad F\geq 0 \qquad (11)$$

Nonlinear Model Equation

In order to solve Eq. (11), we consider the following nonlinear model equation:

$$-[\alpha U_{YY}+\beta U_{XX}]+PU_X+Q_1 UU_Y+R_1 U+R_2 U^2=S_1 \qquad (12)$$

for which we can apply two classical linearization procedures.

Newton-Raphson Linearization (NRL)

Taking the notations used by Blottner,[12] this procedure is such that

$$U^2=2U\bar{U}-\bar{U}^2+O(\epsilon^2)$$

$$UU_Y=\bar{U}U_Y+\bar{U}_Y U-\bar{U}\bar{U}_Y+O(\epsilon\epsilon_Y)$$

where the barred quantities are determined from a previous iteration, $\epsilon=U-\bar{U}$. Then Eq. (12) takes the linear form (6) by setting

$$Q=Q_1\bar{U}, \quad R=R_1+R_2\bar{U}+(Q_1\bar{U}_Y+R_2\bar{U}) \qquad (12a)$$

where R must be positive:

$$S=S_1+(Q_1\bar{U}_Y+R_2\bar{U})\bar{U}$$

Direct Linearization (DL)

Using the following simplier procedure:

$$U^2=\bar{U}U+O(\epsilon), \quad UU_Y=\bar{U}U_Y+O(\epsilon,\epsilon_Y)$$

Eq. (12) takes the linear form (6) by setting

$$Q=Q_1\bar{U}, \quad R=R_1+R_2\bar{U}, \quad S=S_1 \qquad (12b)$$

As the iterative process described in this paper is used until convergence ($\epsilon\to 0$), there is no additional error introduced by these two kinds of linearization procedures. But, as we shall see in Sec. IV, the choice of linearization procedure can influence the conditions of the convergence of the iterative process.

Boundary Conditions Involving Derivatives

We shall also have to consider the linearized equation (6) with the following boundary conditions, involving derivative expressions:

$$U(X_0,Y)=U_0(Y), \quad U(X_{n+1},Y)=U_{n+1}(Y) \qquad (13)$$

$$U_Y(X,Y_0)=0, \quad U_Y(X,Y_{m+1})=0 \qquad (14)$$

Both of conditions (14) can be expressed by second-order accurate formulas. The first one, at $Y=Y_0$, can be written as

$$U_{j,0}=a_1 U_{j,1}+a_2 U_{j,2} \qquad (15)$$

The system of difference equations for the first ray, $k=1$, is then

$$-B_1 U_{j-1,1}+(D_1-a_1E_1)U_{j,1}-C_1U_{j+1,1}-(F_1+a_2E_1)U_{j,2}=S_1$$

in which $D_1=B_1+C_1+E_1+F_1+R_1$, and where D_1 stands for $D_{j,1}$, B_1 for $B_{j,1}$, etc. The expression of conditions (7) in this case is

$$B_1\geq 0, \quad C_1\geq 0, \quad F_1+a_2E_1\geq 0 \qquad (16)$$

The second condition at $Y=Y_{m+1}$ can be written in the same way as

$$U_{j,m+1}=b_1U_{j,m}+b_2U_{j,m-1} \qquad (17)$$

The system of difference equations for the last ray, $k=m$, is

$$-B_mU_{j-1,m}+(D_m-b_1F_m)U_{j,m}-C_mU_{j+1,m}-(E_m+b_2F_m)U_{j,m-1}=S_m$$

in which

$$D_m=B_m+C_m+E_m+F_m+R_m$$

where D_m stands for $D_{j,m}$, B_m for $B_{j,m}$, etc. Thus, the conditions (7) give

$$B_m\geq 0, \quad C_m\geq 0, \quad E_m+b_2F_m\geq 0 \qquad (18)$$

IV. Finite-Difference Schemes

The following finite-difference approximation formulas are given, at a current point (X_j, Y_k), for a variable step size. For the second derivatives, we take

$$U_{XX}=2cDU_{j+1}+2cDU_j \qquad (19)$$

where

$$DU_{j+1}=\frac{U_{j+1,k}-U_{j,k}}{\Delta X_{j+1}}, \quad DU_j=\frac{U_{j-1,k}-U_{j,k}}{\Delta X_j} \qquad (20a)$$

$$\Delta X_{j+1}=X_{j+1}-X_j, \quad c=1/(X_{j+1}-X_{j-1}) \qquad (20b)$$

For the first derivatives, four different kinds of schemes have been discussed and used:

Centered

$$U_X=aDU_{j+1}-bDU_j \qquad (21)$$

where $a=c(X_j-X_{j-1})$, and $b=c(X_{j+1}-X_j)$.

Noncentered Upwind

$$PU_X=pDU_{j+1}+qDU_j \qquad (22)$$

where $p=-\dfrac{|P|-P}{2}$ and $q=-\dfrac{|P|+P}{2}$

Centered Modified Upwind ($n°1$)

$$PU_X=apDU_{j+1}+bqDU_j-(aq\overline{DU}_{j+1}+bp\overline{DU}_j) \qquad (23)$$

where the barred quantities are determined from a previous iteration.

Centered Modified Upwind ($n°2$)

$$PU_X=pDU_{j+1}+qDU_j-(aq+bp)(\overline{DU}_{j+1}+\overline{DU}_j) \qquad (24)$$

These two last schemes have been built following the idea given by Khosla and Rubin,[13] as they have exactly the same truncation error as the centered one.

Step Size

With a constant step size, the formulas (19, 21, 23, and 24) are second-order accurate. With a variable step size, these formulas can be considered again as second-order accurate only if the step-size variation is slow. Variable step size was used in the two directions η and ϕ. In the η direction, the step size is taken shorter at $\eta=0$, where the gradients are stronger. In the ϕ direction, the step size is shorter near $\phi=\pi$, also to account for a severe azimuthal gradient in this region. For these two directions, we choose a geometric progression for the step size $\Delta X_{j+1}=r\Delta X_j$. In this way, X_j is given in a function of the minimum step size ΔX_1 and of the ratio r by

$X_j = \Delta X_l [(r^j - 1)/(r-1)]$. For a given value of the number of points of the grid $n \times m$ and of the ratios X_n/X_l and Y_m/Y_l, it is easy to find the values of r in the X and Y directions.

Convergence Conditions

For the linear equation (6) and the nonlinear equation (12), linearized with the (NRL) or with the (DL) procedures, we can derive sufficient conditions for the convergence of the (J), (LJ), (GS), and (LGS) methods, in terms of the coefficients α, β, P, Q, R, S.

Boundary Conditions of the Type (8) and (9)

In this case, the scheme (21) appears to be only conditionally convergent. The convergence conditions are

$$\alpha_{j,k} \geq \frac{|Q_{j,k}|}{2} \max(\Delta Y_k,\ \Delta Y_{k+1}) \tag{25}$$

$$\beta_{j,k} \geq \frac{|P_{j,k}|}{2} \max(\Delta X_j,\ \Delta X_{j+1}) \tag{26}$$

On the contrary, the three others (22-24) can be shown to be unconditionally convergent. These conditions agree with the ones derived by several other authors for a constant step size.

The convergence conditions (25) and (26) are the same for the (NRL) and (DL) procedures, (12a) and (12b). But, if we consider the parameter R that gives a contribution to the diagonal coefficients D in the difference system (10), we have $R_{NRL} = R_{DL} + R'$, where $R' = Q_l \bar{U}_Y + R_2 \bar{U}$. For the azimuthal momentum equation (1), $R' = (1/U_e)/(\partial W/\partial\theta)$, and thus $R' > 0$ only on the windward side. On the leeward side, $R' < 0$, and the condition $R_{NLR} \geq 0$ could be violated for the large incidences, especially near the wall. In addition, the (NRL) and (DL) procedures have been checked and compared in several numerical experiments in the case of small incidences. The results are practically identical and are obtained with about the same number of iterations. Consequently, preference has been given, in this study, to the (DL) procedure.

Boundary Conditions of the Type (14)

For these boundary conditions, there are additional convergence conditions for all of the schemes (21-24). As an example, conditions (16) can be expressed for the scheme (22) as follows:

$$\alpha_{j,l} \geq \frac{|Q_{j,l}|}{2} \Delta Y_l; \quad j = 1,2,\ldots,n \tag{27}$$

$$\alpha_{j,m} \geq \frac{|Q_{j,m}|}{2} \Delta Y_{m+1}; \quad j = 1,2,\ldots,n \tag{28}$$

For all of the schemes, we can derive some analogous conditions. The condition for $k=1$, like condition (27), generally is not satisfied in our calculations. Indeed, it would be possible to propose a procedure in which this convergence condition does not appear, simply by taking for the boundary condition (14) a first-order expression $[u_0 = f(u_l)]$. But we recall that all of the convergence conditions discussed in the present study are only sufficient, and the calculation trials show that, in fact, this does not disturb the convergence when the "major" conditions (25) and (26) are satisfied. These conditions [(25) and (26)] have been tested in the previous studies[9,11] in which the system (1-4) was considered only in a region very close to $\phi = \pi$. In that case, this system was reduced to a single equation (azimuthal momentum equation) in order to obtain a correction of the solutions of the usual Prandtl equations, which are known to become invalid when $\phi \to \pi$, as mentioned before.

V. Solution of the Governing Equations

The governing equations (1-4) are written in the linearized form (6) by using the direct linearization procedure. Furthermore, they are uncoupled when taking for the coefficients of each equation the value at the previous iteration. These linearized and uncoupled equations are discretized by using the finite-difference formulas (19) and one of the four schemes (21-24), or possibly a combination of two of these schemes in a manner that will be discussed later. The boundary conditions are expressed as in (13). The Line-Gauss-Seidel iterative method is used to solve all of these resulting difference equations successively in the same iteration loop. The iterations are stopped when the solution is everywhere convergent to a given number of figures.

Initial Solution

The experience shows that it is generally less time-consuming to calculate the "parabolic" Prandtl equations and to use the result as an initial solution to start the calculation of the elliptic equations (1-3). These parabolic equations are obtained by setting $\alpha = 0$ and solved by the same iterative process as for the elliptic ones. For these equations, an initial solution also is necessary to start the iterative method. This initial solution must be sufficiently realistic to prevent the breakdown of the iterative procedure. Such a breakdown is due to a certain coefficient of the equations containing the square root of the temperature. (This appears by expressing the viscosity by the Sutherland law in C.) In order to avoid this breakdown we use the Prandtl equations, at $\phi = 0$, simplified by taking $C = 1$ and by using the Crocco temperature velocity relationship. After obtaining profiles of u, $w_0 = w/K'$, and T at $\phi = 0$, the initial (guessed) solution for every ϕ is simply written:

$$u(\phi) = u(\phi = 0), \quad w(\phi) = w_0 K', \quad T(\phi) = T(\phi = 0)$$

Combined Schemes

The noncentered scheme, previously used for a single equation (9) or (11), appeared to be unconditionally stable for all of the *Rex* values experimented with in the present study. The results show that the "elliptic" solution coincides with the "parabolic" one over most of the cone that we call the domain of "parabolicity." These two solutions are different only in the vicinity of $\phi = \pi$, in a region that we call domain of "ellipticity." To obtain accurate solutions in this domain, and because of the convergence requirements, a combination of two schemes was used occasionally. As an example, the noncentered scheme is used in the largest part of the domain of "parabolicity" and the centered one in the complementary region, including just the domain of "ellipticity." This last combination is like the one proposed by Spalding.[14]

Computational Indications

In order to save computational time, as in Ref. 5, the iteration logic has been structured in such a manner that, when the solution along a ϕ line converges to the desired number of figures, that line is dropped from the iteration loop. A computer program was developed for the CII.10070 computer. For 40 points in the η direction and 58 points in the ϕ direction, and for optimized values of $\Delta\phi_{\min}$ and $\Delta\eta_l$, the computation of both parabolic and elliptic solutions requires about 7 min for a given value of x and for a required precision of four figures.

VI. Numerical Results

Parabolic Solution

In this case, $\alpha = 0$, and thus the convergence condition (25) for the centered scheme never is satisfied, and this scheme effectively fails. On the contrary, the noncentered scheme, which consists here of a purely implicit scheme in the ϕ direction, always is convergent.

The scheme (24) also was used tentatively but without success, probably because of the downstream influence involved in the right-hand side of the difference equations, which cannot be supported by the parabolic equations when the boundary $\phi=\pi$ is reached. However, experiment has shown that this scheme can be used in the greatest part of the cone, except for the last ϕ steps, when $\phi-\pi$, where the noncentered scheme must be used. The results obtained by these combined schemes and by the noncentered scheme agree very well with the results obtained previously with the Crank-Nicolson scheme.[2] The influence of the step size on the accuracy has been tested. Figures 2 and 3 show that the grid spacing must be refined greatly in the vicinity of $\phi=\pi$, which imposes the use of a variable spacing.

Elliptic Solution

Choice of $\Delta\phi_{min}$

In order to obtain more accurate results in the domain of "ellipticity," the calculations were performed first following the noncentered scheme by reducing the size of the last step near $\phi=\pi$, $\Delta\phi_{\min}$. But, contrary to the expectation, the results obtained for a required precision, say $\epsilon=2\times10^{-4}$, do not converge to a limiting value when $\Delta\phi_{\min}\rightarrow 0$ (Fig. 4). The trend of this curve shown in Fig. 4, when $\Delta\phi_{\min}\rightarrow 0$, is not due, as can be checked, to roundoff error effect. In fact, the rate of convergence decreases with $\Delta\phi_{\min}$, and thus the number of iterations increases when $\Delta\phi_{\min}\rightarrow 0$ (Fig. 5). This has been shown already by Isaacson and Keller[15] for the Laplace equation. And it appears incorrect to limit the number of iterations to the same figure, independently of the value of the rate of convergence. Indeed, the additional results obtained for different ϵ and given in Fig. 4 show that it is necessary to reduce the required precision down to $\epsilon=6\times10^{-5}$ to obtain a correct solution for $\Delta\phi_{\min}\gtrsim 0.25°$, when $Rex=2\times10^{5}$. Obviously, the closer $\Delta\phi_{\min}$ approaches to zero, the smaller ϵ should be chosen. A behavior similar to that just described has been observed for each value of Rex within the range considered in this investigation, but it is not observed for a fixed value of $\Delta\phi_{\min}$ but rather for a roughly constant value of $\Delta\tilde{\phi}_{\min}=Rex^{1/2}\Delta\theta_{\min}$ (Fig. 6).

As will be shown later, the azimuthal thickness of the domain of "ellipticity," $\pi-\phi_E$, varies approximately as $Rex^{-1/2}$. It appears probable (although not demonstrated) that a parameter caracterizing the rate of convergence is the ratio $\Delta\tilde{\phi}_{\min}/(\pi-\phi_E)$, at least for Rex within the range considered in this investigation.

Remark Concerning the Scheme (23)

The scheme (23), which is theoretically unconditionally convergent, fails, except for Rex very small. But the scheme (24), which is built with the same general idea, was found to converge for all considered values of Rex. It can be observed that these schemes contain in the right-hand side of the difference equations some downstream influence that could not be supported always by Eqs. (1-4) in the domain of parabolicity. Indeed, if the true solution is approached (for which $P>0$ and thus $p=0$), it follows that

$$U_\theta=-bDU_k+a\overline{DU}_{k+1}$$

for the scheme (23), and

$$U_\theta=-DU_k+a(\overline{DU}_{k+1}+\overline{DU}_k)$$

for the scheme (24). Consequently, this supported downstream effect appears to be ponderated in the scheme (24). This could explain partly why this scheme, which will be used, works much better than the scheme (23).

Comparison Between the Different Schemes

When the minimum step size $\Delta\tilde{\phi}_{\min}$ is optimized [that is, roughly, when $\Delta\tilde{\phi}_{\min}$ is at least a tenth of $(\pi-\phi_E)$], the results

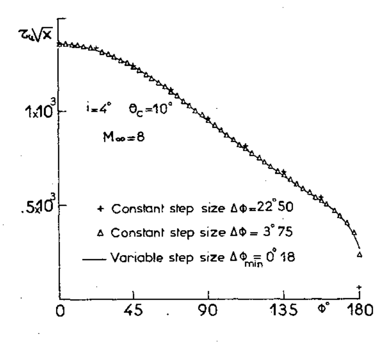

Fig. 2 Azimuthal variation of the skin friction (parabolic case).

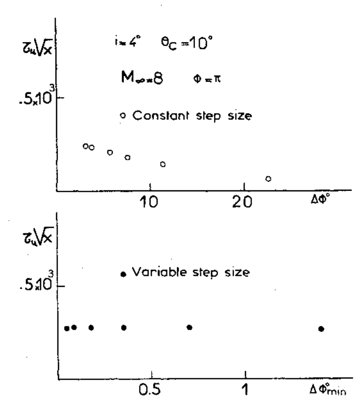

Fig. 3 Skin friction in the leeward plane (parabolic case).

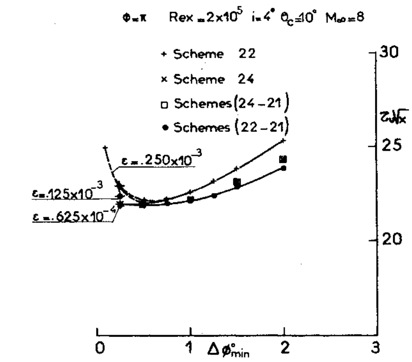

Fig. 4 Skin friction in the leeward plane.

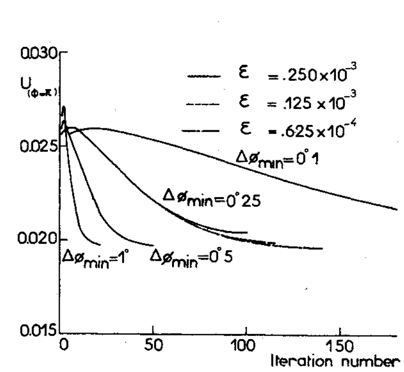

Fig. 5 Rate of convergence of the solution in the leeward plane.

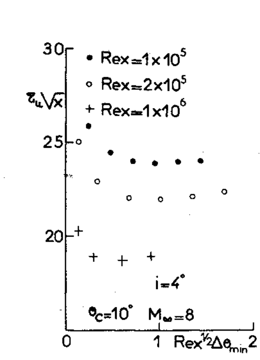

Fig. 6 $\Delta\tilde{\phi}_{\min}$ effect on skin friction.

obtained with each scheme are practically identical (Fig. 4). When $\Delta\phi_{min}$ is increased (the solutions become less and less accurate), the discrepancy between the results obtained by the different schemes increases. This discrepancy is most important for the noncentered scheme, which is only first-order accurate. The second-order-accurate scheme (24) could be preferred, as it appears as unconditionally stable as the noncentered one.

Results

Several calculations have been made for the case of a 10° half-angle cone already considered in previous theoretical[1-6] and experimental[7] investigations. Up to the present, these calculations have been carried out only for incidences not too large in order to exclude the occurrence of separated flow. The results obtained for the incidence $I=4°$ and concerning the heat-transfer rate q_p and the longitudinal skin friction τ_u at the wall are given in Fig. 7. More precisely, Fig. 7 shows the azimuthal variation of $q_p(x)^{1/2}$ and $\tau_u(x)^{1/2}$, as they do not depend upon x on the greatest part of the cone (i.e., in the domain of parabolicity), where the solutions are self-similar. In the same way, Fig. 8 shows that variation of $\delta(x)^{-1/2}$, where δ is the displacement thickness of the boundary layer. These last two figures confirm that the second azimuthal derivatives take significant value when $\phi\to\pi$. Figure 9 shows the azimuthal variation of the elliptic values of $q_p(x)^{1/2}$ in terms of Rex compared to the "parabolic" ones, which are independent of x. The comparison of these results allows the determination of the thickness of the domain of ellipticity at the wall. This domain is defined here as bounded by the generator ϕ_E beyond which the difference between these parabolic and elliptic results exceeds 1%. This thickness, $\pi-\phi_E$, as shown in Fig. 10, decreases when Rex is increased, behaving roughly as $Rex^{-1/2}$. This seems to indicate, as suggested by Krause,[16] the use for the governing equations (1-4) of a stretching of the azimuthal coordinate, $\bar{\phi}=\theta\, Rex^{1/2}$, in order to study only the ellipticity region. This interesting approach was not used up to the present, as we were intending instead to develop a method enabling us to extend this study further to the case of governing equations much more complete than Eq. (1-4) (which contain the simplifications discussed in Sec. II). The effect of x on $q_p(x)^{1/2}$ and $\delta/(x)^{1/2}$ is shown in Figs. 11 and 12, in the form used by Lin and Rubin.[4] These figures confirm that the boundary layer is similar over most of the cone and nonsimilar when $\phi\to\pi$.

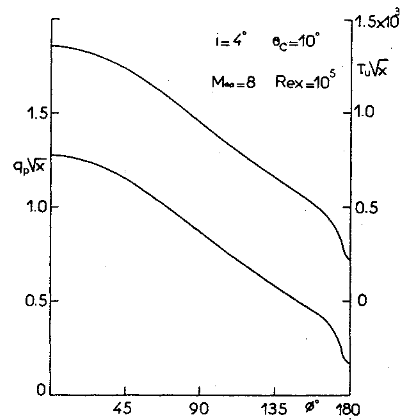

Fig. 7 Heat transfer and skin friction.

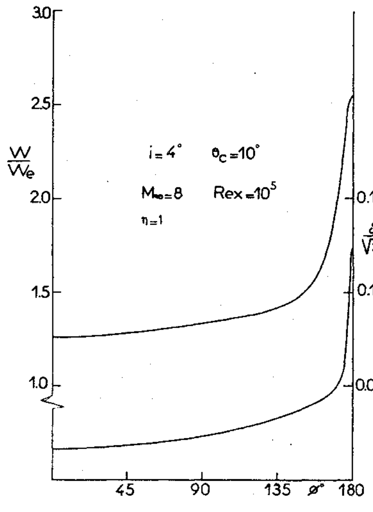

Fig. 8 Azimuthal velocity and displacement thickness.

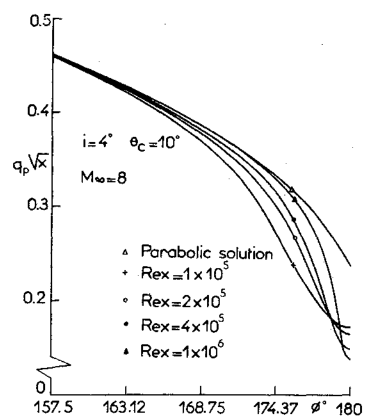

Fig. 9 Heat transfer (*Rex* effect).

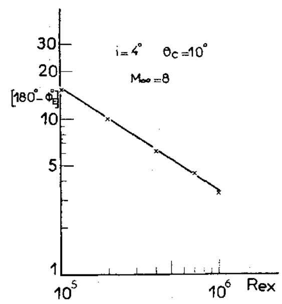

Fig. 10 Thickness of the domain of "ellipticity."

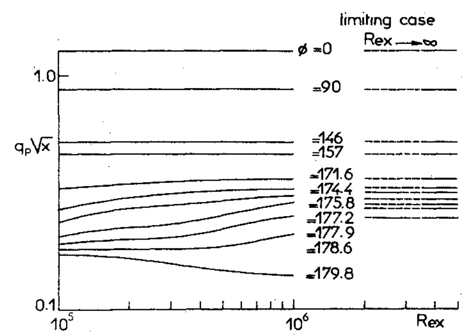

Fig. 11 Heat transfer

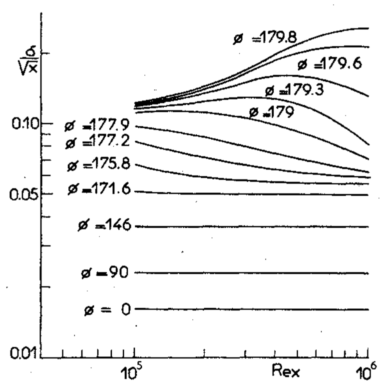

Fig. 12 Displacement thickness.

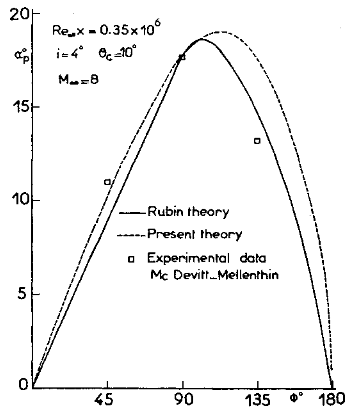

Fig. 13 Surface streamline angle.

The results concerning the limiting streamline angle at the wall, α_p (Fig. 13) do not agree very well with those given by Lin and Rubin.[4] The present results seem to give an exaggerated value of α_p from $\phi = 110°$ to $\phi = 180°$. The same prediction has been found for a 9° half-angle cone at $I=5°$, for which comparisons have been made with the experiments of Marcillat (Fig. 2 of Ref. 8). For this last case, the new theoretical prediction of α_p is identical to the one obtained previously[8] by means of the Prandtl equations, except when $\phi \to \pi$, where the new values of α_p reach zero. This discrepancy with both experimental results[8] and more accurate theoretical predictions[4] seems to indicate that the boundary layer on the cone is not yet sufficiently well modeled by Eqs. (1-4), which assume similarity and in which we have retained only the second azimuthal derivatives. We plan to investigate further the influence of such assumptions. Indeed, the method presented in this study to control the convergence of the finite-difference schemes can be extended easily to the case of non-similar equations involving first-order derivatives in the longitudinal direction, as in the Lin and Rubin[4] analysis.

VII. Conclusions

A method for solving the difference equations describing the boundary layer (which involve azimuthal diffusion terms) around a cone at incidence has been analyzed. The following points have been underlined:

1) Some conditions sufficient for the convergence are derived for a class of "difference system" having the properties of what we have called *V*-matrices.

2) For such difference equations, we obtain the same conditions sufficient for the convergence of the (J), (LJ), (GS), and (LGS) iterative methods, respectively. In addition, the (LGS) method is shown to require less computing time.

3) These conditions for the classical centered and non-centered upwind schemes rejoin the ones given by several other authors for a constant step size.

4) Two centered modified upwind schemes, after the one proposed by Khosla and Rubin,[13] have been proposed and shown to be second-order accurate and unconditionally stable, but only the second one was found to converge for any of the *Rex* values considered in this study.

5) In the greatest part of the cone (called domain of "parabolicity"), the governing equations behave as the parabolic Prandtl equations.

6) The complementary region (domain of "ellipticity") is shown to fill a region, $\pi - \phi_E$, which behaves with x roughly as $Rex^{-1/2}$.

7) A reduced step size must be specified in this domain of "ellipticity," and attention must be devoted to adjust the required precision ϵ adequately.

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